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ABSTRACT

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The countdown is a large program which consists of the logical combination of several activities, with various recycle policies in the event of failures. Each activity is a stochastic process, in that the time required for its performance may be a random variable. In addition, the occurrence of equipment failures constitutes a random process. The entire countdown is thus a stochastic process and the total time required to complete it is a random variable. The probability density function for the total countdown duration is derived in terms of the distributions of the basic random variables for each activity.

The model which is analyzed is quite general and is descriptive of many programs which consist of the combination of several individual activities. The analysis procedures are applicable to many other similar problems.

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SUBJECT: The Analysis of a Countdown as a Stochastic Sequential Process with Recycle Policies - Case 130

DATE: August 16, 1965

FROM: J. S. Engel

TM-65-1031-2

TECHNICAL MEMORANDUM

1. Introduction

This memorandum presents a method for analyzing large programs which consist of the logical combination of several activities. Each activity is a stochastic process, in that the time required for its performance may be a random variable. In addition, during the performance of an activity, equipment failures may occur, at random times. In the event of an equipment failure, various recycle policies may be pursued. After the failure has been repaired, the activity may be continued from the point of failure, the individual activity may have to be reinitiated, or the process may be required to recycle to some prior point in the program, so that some completedactivities may have to be reperformed. The entire program is thus a stochastic process, and the total time required to complete the program is a random variable. The analysis consists of determining the probability density function (or cumulative distribution function) for this random variable, the total time required to complete the entire program, given the distributions of the basic random variables for each activity. The distribution may also be presented implicitly, in the form of the characteristic function.

The particular problem which has prompted this study is the analysis of the countdown process, and the model which is considered in this memorandum was generated to represent that particular process. However, the model is quite general and is descriptive of many programs which consist of the combination of several individual activities. The analysis procedures which are presented here may be applied to many other similar problems.

A recent Technical Memorandum[1] has discussed this problem and has described a method of solution utilizing a Monte Carlo simulation on a digital computer. The Monte Carlo tech-

[1] "Status Report on Countdown Simulation-Case 140", E. B. Parker III and P. S. Schaenman, December 17, 1964, TM-64-1031-3.

nique consists of selecting a single value for each of the basic random variables from an appropriate distribution and combining them in the proper manner in order to determine a single value for the total time required to complete the countdown. If this is done many times and a histogram of the resulting values is constructed, the probability density function for the time to complete the countdown may be "experimentally" obtained.

The procedures described in this memorandum comprise an analytic technique for deriving the distribution for the total time as a function of the individual component distributions. The end result is a mathematical expression for the probability density function (or the cumulative distribution function) of the time to complete the countdown. The evaluation of this expression involves a series of integrations and convolutions and may, for complicated cases, have to be performed using numerical methods on a digital computer. The distinction, however, is that the computer is then used as a calculating machine; it is not programmed to simulate the operation. In addition, the computer is not always required for the evaluation of the expression. For many cases, the integrations and convolutions may be performed symbolically.

The general problem may be described as follows. The program consists of a collection of individual activities which are logically interrelated. Certain of the activities may be performed concurrently. Others must be performed serially; that is, the completion of certain activities is necessary before certain others may be initiated. These logical interrelationships may be represented by a "PERT-like" network. The deviation from the normal PERT representation results from the introduction of recycles. During the performance of an activity, an equipment failure may occur. Depending on the nature of the activity being performed and on the nature of those activities already completed, the failure may merely introduce a "hold"; it may require that the individual activity be reinitiated; or it may necessitate recycling to a prior point and repeating some of the completed activities. The problem is to "reduce" a complex network, including recycle paths, to a single branch or composite activity, representing the entire countdown, and to determine the distribution of the times to complete this composite. This is done in a sequence of steps. Small groups of two or more activities are combined to form composite activities. Then, small groups of composite activities are combined in the same manner. The procedure is repeated until all activities have been combined into a single composite. This memorandum presents the techniques for combining activities which have various logical interrelationships.

2. Summary

In Section 3, the single activity is discussed. An individual activity may be described statistically by three basic independent random variables. The first of these is the time required to perform the activity when the effects of failures are excluded and is denoted T_o. The second random variable is the time between failures of the equipment used in performing the activity and is denoted T_f. The third is the time required to locate and repair the failure so that the process may be recycled and is denoted T_r. It is assumed that the distributions of these random variables are either known or well estimated. From these three distributions, the characteristic function for the total time required to complete the activity, including the effects of failures and recycles, is derived. There are two possible recycle policies, and both are investigated. When the activity is "held" during each repair and then continued from the point at which the failure had occurred, the total time required to complete the activity, denoted T_A, has a characteristic function given by:

$$M_{T_A}(\omega) = M_{T_o}(\omega) M_{T_R}(\omega) \qquad (2\text{-}1)$$

where the characteristic function $M_{T_R}(\omega)$ is given by:

$$M_{T_R}(\omega) = \frac{M_{T_r}(\omega)}{2\pi} \int_{-\infty}^{\infty} \frac{[1-M_{T_f}(x)]M_{T_f}(x)M_{T_o}(-x)}{jx[1-M_{T_r}(\omega)M_{T_f}(x)]}\, dx \qquad (2\text{-}2)$$

When the activity is reinitiated after each failure, the total time required to complete the activity has a characteristic function given by:

$$M_{T_A}(\omega) = \frac{(1-p)M_{T_o'}(\omega)}{1-p\, M_{T_f'}(\omega)\, M_{T_r}(\omega)} \qquad (2\text{-}3)$$

where the probability p is equal to:

$$p = \int_0^\infty F_{T_f}(t_o) f_{T_o}(t_o) dt_o \tag{2-4}$$

the characteristic function $M_{T'_o}(\omega)$ is the Fourier transform of the probability density function given by:

$$f_{T'_o}(t'_o) = \frac{1}{1-p} f_{T_o}(t'_o)\,[1-F_{T_f}(t'_o)] \tag{2-5}$$

and the characteristic function $M_{T'_f}(\omega)$ is the Fourier transform of the probability density function given by:

$$f_{T'_f}(t'_f) = \frac{1}{p} f_{T_f}(t'_f)\,[1-F_{T_o}(t'_f)] \tag{2-6}$$

The procedures for reducing two successive activities to a single composite activity are derived in Section 4. There are two possible recycle policies, and both of these are considered. When the recycle policies for the two activities, denoted activity A and activity B, are independent so that a failure in the second activity only affects that activity, the characteristic function for the time to complete the composite activity, denoted T_C, is given by:

$$M_{T_C}(\omega) = M_{T_A}(\omega) M_{T_B}(\omega) \tag{2-7}$$

When the recycle policies are dependent and a failure during the second activity, activity B, requires that the first activity, activity A, be reinitiated, the characteristic function for the time to complete the composite activity is given by:

$$M_{T_C}(\omega) = \frac{(1-p_B)M_{T_A}(\omega)M_{T'_{oB}}(\omega)}{1-p_B M_{T'_{fB}}(\omega)M_{T_{rB}}(\omega)M_{T_A}(\omega)} \qquad (2\text{-}8)$$

In Section 5, procedures are derived for combining two parallel activities, performed concurrently. There are six possible recycle policies, and these are all considered. When the recycle policies are independent, so that a failure in either activity affects only that activity, the time to complete the composite activity has a probability density function given by:

$$f_{T_C}(t_C) = f_{T_A}(t_C)F_{T_B}(t_C) + F_{T_A}(t_C)f_{T_B}(t_C) \qquad (2\text{-}9)$$

and a characteristic function given by:

$$M_{T_C}(\omega) = \frac{\omega}{2\pi}\int_{-\infty}^{\infty} \frac{M_{T_A}(x)M_{T_B}(\omega-x)}{jx(\omega-x)}\,dx \qquad (2\text{-}10)$$

There are two possible semi-dependent recycle policies. When a failure in activity A requires that both activities be held during repairs and then continued, while a failure in activity B does not interrupt activity A, the characteristic function for the time to complete the composite activity is given by:

$$M_{T_C}(\omega) = \frac{M_{T_{RA}}(\omega)}{2\pi}\int_{-\infty}^{\infty} \frac{M_{T_{oA}}(x)M_{T_B}(\omega-x)}{jx(\omega-x)}\,dx \qquad (2\text{-}11)$$

When a failure in activity A requires that both activities be reinitiated, while a failure in activity B does not interrupt activity A, the characteristic function for the time to complete the composite activity is given by:

$$M_{T_C}(\omega) = \frac{(1-p_A)M_{T_S}(\omega)}{1-p_A M_{T'_{fA}}(\omega)M_{T_{rA}}(\omega)} \qquad (2\text{-}12)$$

where the characteristic function $M_{T_S}(\omega)$ is given by:

$$M_{T_S}(\omega) = \frac{\omega}{2\pi}\int_{-\infty}^{\infty} \frac{M_{T'_{oA}}(x)M_{T_B}(\omega-x)}{jx(\omega-x)}\,dx \qquad (2\text{-}13)$$

where the probability p'_A is equal to:

$$p'_A = p_A\left[(1-p_B) + p_B\int_0^{\infty} F_{T_{fA}}(t_{fB})f_{T_{fB}}(t_{fB})dt_{fB}\right] \qquad (2\text{-}17)$$

and the probability p'_B is equal to:

$$p'_B = p_B\left[(1-p_A) + p_A\int_0^{\infty} F_{T_{fB}}(t_{fA})f_{T_{fA}}(t_{fA})dt_{fA}\right] \qquad (2\text{-}18)$$

When a failure in activity A requires that both activities be reinitiated after repairs, while a failure in activity B requires that both activities be held during repairs and then continued, the characteristic function for the time required to complete the composite is given by:

$$M_{T_C}(\omega) = \frac{(1-p_A)M_{T_{RB}}(\omega)M_{T_S}(\omega)}{1-p_A M_{T'_{fA}}(\omega)M_{T_{rA}}(\omega)}. \qquad (2\text{-}19)$$

Section 6 considers the non-series-parallel combination of activities. There are an infinite number of such combinations which are possible, and a general method for reducing them to a single composite activity is presented. The reduction of one particular non-series-parallel combination is shown as an example of the technique. This general method may be extended to other such combinations.

3. Single Activity

The basic building block for the entire process is the single activity. Excluding the effects of failures, the activity takes a time T_o to perform. This time may be a random variable with a known, or well estimated, probability density function $f_{T_o}(t_o)$. Before the activity has been completed, however, a failure may occur in the equipment used in performing the activity. It is assumed that the reliability of the equipment is known in the sense that the time between failures is a random variable T_f with known probability density function $f_{T_f}(t_f)$. If a failure should occur, the failed item is repaired, taking a time T_r, which may be a random variable. For the total set of equipment used in performing the activity, the probability density function $f_{T_r}(t_r)$ is assumed to be known or well estimated. It should be noted that these density functions are considered as generalized functions. In particular, they may include Dirac delta functions.

There are two possible recycle policies in the event of a failure. After the failed item has been repaired, the activity may be continued from the point at which the failure occurred, or it may be reinitiated. Let the random variable T_A be defined as the total time required to complete the activity, taking failures and repairs into account. The problem, then, is to find the probability density function $f_{T_A}(t_A)$, in terms of the known probability density functions $f_{T_o}(t_o)$, $f_{T_f}(t_f)$; and $f_{T_r}(t_r)$.

3.1 Single Activity with "Hold and Continue" Policy

When the activity is "held" during each repair and then continued from the point at which the failure occurred, the total time required to complete the activity is equal to:

$$T_A = T_o + T_R \tag{3-1}$$

where T_R is equal to the sum of all the repair times T_r required. Let p_i be defined as the probability that i or more failures occur while performing the activity:

$$p_i \triangleq \Pr [\text{i or more failures}] \tag{3-2}$$

This is equal to the probability that the sum of i independent values of the random variable T_f is less than T_o:

$$p_i = \Pr [T_{f_1} + T_{f_2} + \ldots + T_{f_i} < T_o] \tag{3-3}$$

and this is given by:

$$p_i = \int_o^\infty F_{T_{f_1} + T_{f_2} + \ldots + T_{f_i}}(t_o) dF_{T_o}(t_o)$$

$$p_i = \int_o^\infty F_{T_{f_1}+T_{f_2}+\ldots+T_{f_i}}(t_o) f_{T_o}(t_o)\, dt_o \tag{3-4}$$

Where $F_{T_f}(t_f)$ and $F_{T_o}(t_o)$ are the cumulative distribution functions for T_f and T_o, respectively. Equation (3-4) is equivalent to:

$$p_i = \int_o^\infty \left\{ \int_o^{t_o} [\text{i-fold convolution of } f_{T_f}(t_f) \text{ with itself}] dt_f \right\} f_{T_o}(t_o) dt_o$$

By Parseval's theorem, this is equal to:

$$p_i = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{j\omega} \quad M_{T_f}^{i} (\omega) \; M_{T_o} (-\omega) \; d\omega \qquad (3\text{-}5)$$

where $M_{T_f}(\omega)$ and $M_{T_o}(\omega)$ are the characteristic functions of T_f and T_o, respectively, and are equal to the Fourier transforms of their respective probability density functions.

Given the condition that exactly one failure occurs, which has a probability $p_1 - p_2$, T_R is equal to T_r and has a probability density function equal to $f_{T_r}(t_R)$. Given the condition that exactly two failures occur, which has a probability $p_2 - p_3$, T_R is equal to $T_{r_1} + T_{r_2}$ and has a probability density function equal to $f_{T_r}(t_R) * f_{T_r}(t_R)$, where the asterisk denotes convolution. Summing over all the possible numbers of failures, multiplied by their respective probabilities of occurrence, yields the probability density function for T_R:

$$f_{T_R}(t_R) = \sum_{i=1}^{\infty} (p_i - p_{i+1}) \text{ [i-fold convolution of } f_{T_r}(t_R) \text{ with itself]} \qquad (3\text{-}6)$$

Taking the Fourier transform of both sides of equation 3-6 with respect to t_R yields the characteristic function for T_R:

$$M_{T_R}(\omega) = \sum_{i=1}^{\infty} (p_i - p_{i+1}) \; M_{T_r}^{i}(\omega) \qquad (3\text{-}7)$$

Substituting equation 3-5 for p_i and p_{i+1} into equation 3-7 yields:

$$M_{T_R}(\omega) = \sum_{i=1}^{\infty} M_{T_r}^{i}(\omega) \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{jx} \left[M_{T_f}^{i}(x) - M_{T_f}^{i+1}(x) \right] M_{T_o}(-x)dx$$

The order of the integration and summation may be interchanged, yielding:

$$M_{T_R}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{jx} \left[1 - M_{T_f}(x) \right] M_{T_o}(-x) \sum_{i=1}^{\infty} \left[M_{T_r}(\omega) M_{T_f}(x) \right]^{i} dx \tag{3-8}$$

The characteristic function for a random variable has an absolute value bounded by one, and therefore, the power series in equation 3-8 may be written in closed form, yielding:

$$M_{T_R}(\omega) \quad \frac{M_{T_r}(\omega)}{2\pi} \int_{-\infty}^{\infty} \frac{\left[1 - M_{T_f}(x)\right] M_{T_f}(x) M_{T_o}(-x)}{jx \left[1 - M_{T_r}(\omega) M_{T_f}(x)\right]} dx \tag{3-9}$$

The characteristic function for T_A is equal to:

$$M_{T_A}(\omega) = M_{T_o}(\omega) M_{T_R}(\omega) \tag{3-10}$$

The probability density function $f_{T_A}(t_A)$ may be obtained by taking the inverse Fourier transform of equation 3-10. However, subsequent operations involved in combining activities are often in terms of the characteristic functions of the random variables, so that the results may be retained in this form.

3.2 Single Activity with "Reinitiate" Policy

When the activity is reinitiated after each failure, the failures and subsequent repairs are assumed to constitute a renewal process. After each repair the process begins again. The time until the next failure has the same probability density function $f_{T_f}(t_f)$. The attempt to perform the activity is reinitiated, and the time to perform it has the same probability density function $f_{T_o}(t_o)$.

The process described above may be represented by a flow graph of sorts, as shown below, in Figure 3-1:

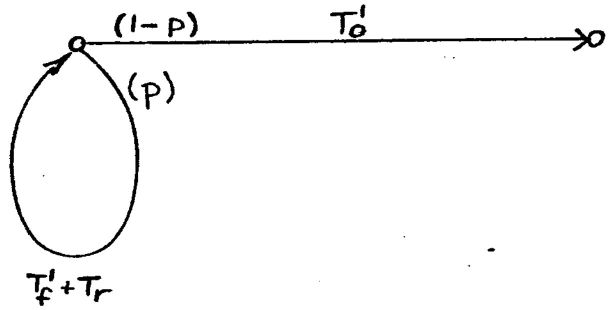

Figure 3-1

The subscripted "T" associated with each path denotes the time to complete that path. The parenthesized expression at each path out of a node denotes the probability of taking that path. The "primes" which distinguish T_o' and T_f' in Figure 3-1 from T_o and T_f should be noted. These will be explained shortly.

For any given attempt to perform the activity, the random variable T_f may be smaller than the random variable T_o, in which case a failure is said to occur. The probability of failure, denoted p, is thus defined by:

$$p \triangleq Pr\ [T_f < T_o] \tag{3-11}$$

and this is equal to:

$$p = \int_o^\infty F_{T_f}(t_o) dF_{T_o}(t_o) = \int_o^\infty F_{T_f}(t_o) f_{T_o}(t_o) dt_o \tag{3-12}$$

It is desired that the flow graph of Figure 3-1 be reduced to a single path, as shown below, in Figure 3-2:

Figure 3-2

where T_A is the total time required to complete the activity, with all possible failures and recycles taken into account. Before deriving the expression for $f_{T_A}(t_A)$, however, it is necessary to define the random variables T_o' and T_f' and to derive their respective probability density functions.

The random variable T_o has been defined as the time to perform the activity, excluding the effects of failure. It is the time required to perform all the actions associated with the activity, not including any actions required as a result of failure. This has a cumulative distribution function defined by:

$$F_{T_o}(t_o) \triangleq Pr\ [T_o \leq t_o] \tag{3-13}$$

The random variable T_o' is the time to complete the forward or "success" path of the flow chart of Figure 3-1. The set of times T_o' is that subset of the times T_o for which the attempt to perform the activity is successful; it is the subset consisting of those times T_o which are less than the associated T_f. The cumulative distribution function for T_o' is therefore the conditional probability:

$$F_{T_o'}(t_o') = \Pr\,[T_o \leq t_o' \mid T_o < T_f] \qquad (3\text{-}14)$$

By the definition of conditional probability, this is equal to:

$$F_{T_o'}(t_o') = \frac{\Pr[T_o \leq t_o' \; ; \; T_o < T_f]}{\Pr[T_o < T_f]}$$

$$F_{T_o'}(t_o') = \frac{\Pr[T_o \leq t_o'] - \Pr[t_o' \geq T_o > T_f]}{\Pr[T_o < T_f]}$$

Substituting the definitions given in equations 3-11 and 3-13 yields:

$$F_{T_o'}(t_o') = \frac{1}{1-p}\left\{F_{T_o}(t_o') - \int_0^{t_o'} [F_{T_o}(t_o') - F_{T_o}(t_f)]\,dF_{T_f}(t_f)\right\}$$

$$F_{T_o'}(t_o') = \frac{1}{1-p}\left\{F_{T_o}(t_o')[1-F_{T_f}(t_o')] + \int_o^{t_o'} F_{T_o}(t_f)f_{T_f}(t_f)dt_f\right\} \tag{3-15}$$

The probability density function $f_{T_o'}(t_o')$ is obtained by differentiating equation 3-15 with respect to t_o'. After rearranging terms, this is equal to:

$$f_{T_o'}(t_o') = \frac{1}{1-p}f_{T_o}(t_o')[1-F_{T_f}(t_o')] \tag{3-16}$$

The random variable T_f has been defined as the time between equipment failures. Its cumulative distribution function:

$$F_{T_f}(t_f) \triangleq Pr[T_f \le t_f] \tag{3-17}$$

is a property of the equipment and is independent of the activity for which the equipment is being used. The random variable T_f' is the time to failure for only those failures which occur before the activity is completed. Failures occurring after the activity is completed are not encountered. The set of times T_f' is thus the subset of those time T_f which are less than the associated T_o. The cumulative distribution function for T_f' is therefore the conditional probability:

$$F_{T_f'}(t_f') = Pr[T_f \le t_f' | T_f < T_o] \tag{3-18}$$

By the definition of conditional probability, this is equal to:

$$F_{T_f'}(t_f') = \frac{Pr[T_f \leq t_f' \; ; \; T_f < T_o]}{Pr[T_f < T_o]}$$

$$F_{T_f'}(t_f') = \frac{Pr[T_f \leq t_f'] - Pr[t_f' \geq T_f > T_o]}{Pr[Tf < T_o]}$$

Substituting the definitions given in equations 3-11 and 3-18 yields:

$$F_{T_f'}(t_f') = \frac{1}{p}\left\{ F_{T_f}(t_f') - \int_o^{t_f'} [F_{T_f}(t_f') - F_{T_f}(t_o)] dF_{T_o}(t_o) \right\}$$

$$F_{T_f'}(t_f') = \frac{1}{p}\left\{ F_{T_f}(t_f')[1 - F_{T_o}(t_f')] + \int_o^{t_f'} F_{T_f}(t_o) f_{T_o}(t_o) dt_o \right\} \qquad (3\text{-}19)$$

Differentiating equation 3-19 with respect to t_f' and rearranging terms yields the conditional probability density function:

$$f_{T_f'}(t_f') = \frac{1}{p} f_{T_f}(t_f') \; [1 - F_{T_o}(t_f')] \qquad (3\text{-}20)$$

Given the condition that no failures occur, which has a probability (1-p), T_A is equal to T_o' and has a probability density function $f_{T_o'}(t_A)$. Given the condition that exactly one failure occurs, which has a probability p(1-p), T_A is equal to $T_o' + T_f' + T_r$ and has a probability density function equal to $f_{T_o'}(t_A)*f_{T_r'}(t_A)*f_{T_r}(t_A)$, where the asterisk denotes convolution. Summing over all the possible numbers of failures, multiplied by their respective probabilities of occurrence, yields the probability distribution function for T_A:

$$f_{T_A}(t_A) = (1-p)\sum_{i=0}^{\infty} p^i f_{T_o'}(t_A) * \left\{ \begin{array}{l} \text{i-fold convolution of } [f_{T_f'}(t_A)*f_{T_r}(t_A) \\ \text{with itself} \end{array} \right\} \tag{3-21}$$

Taking the Fourier transform of both sides of equation 3-21 with respect to t_A yields the characteristic function for T_A:

$$M_{T_A}(\omega) = (1-p)\sum_{i=0}^{\infty} M_{T_o'}(\omega)[p\, M_{T_f'}(\omega)M_{T_r}(\omega)]^i \tag{3-22}$$

The characteristic function of a random variable has an absolute value which is bounded by one. The probability p is also bounded by one. The power series of equation 3-22 may thus be written in the closed form:

$$M_{T_A}(\omega) = \frac{(1-p)M_{T_o'}(\omega)}{1-p\, M_{T_f'}(\omega)M_{T_r}(\omega)} \tag{3-23}$$

4. Serial Succession

4.1 Independent Recycle Policies

The simpler type of serial succession is one with independent recycle policies. Activity B follows activity A and is not initiated until activity A is completed. Should a failure occur during activity B, only that activity is affected. After the failure has been repaired, activity B may be reinitiated or it may be continued from the point at which the failure had occurred. A failure in activity B does not require that activity A reperformed. If the composite of activities A and B is denoted activity C, then the time to complete activity C is equal to the sum:

$$T_C = T_A + T_B \tag{4-1}$$

where T_A and T_B are the total times required to perform the individual activities, with all possible failures taken into account. The characteristic function for T_C is therefore equal to the product:

$$M_{T_C}(\omega) = M_{T_A}(\omega) M_{T_B}(\omega) \tag{4-2}$$

4.2 Dependent Recycle Policies

A complicating modification of the serial succession of two activities occurs when a failure during the second activity results in a reinitiation of the first activity. This recycle policy is shown below, in Figure 4-1:

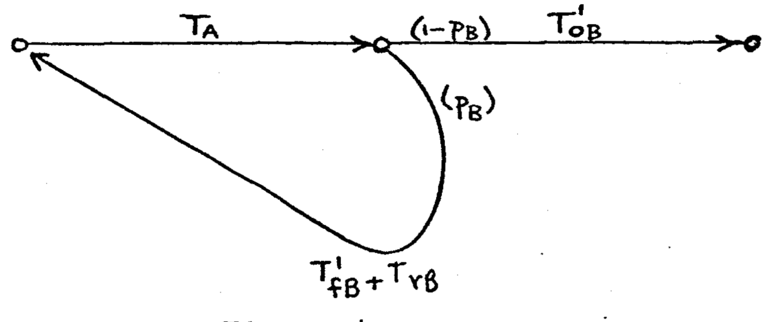

Figure 4-1

where T'_{oB}, T'_{fB}, T_{rB}, and p_B have the meanings previously given for T'_o, T'_f, T_r, and p, respectively. The added subscript "B" denotes the activity to which they refer. Activity A is completed, taking time T_A. Note that this time includes the effects of failures. Then, activity B is initiated; each time a failure occurs, T'_{fB} has elapsed, and T_{rB} plus T_A must be completed before T'_o can begin. The flow graph may be redrawn as shown below, in Figure 4-2:

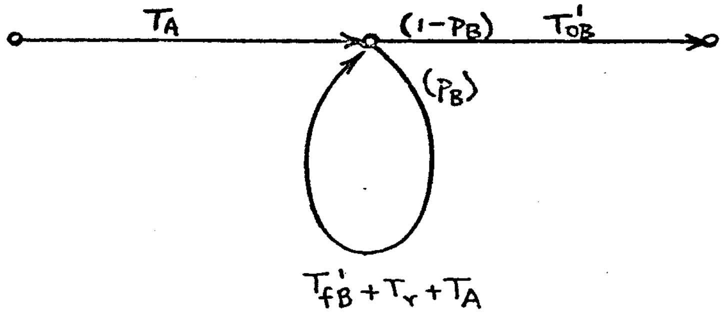

Figure 4-2

In a manner which exectly parallels the reasoning of Section 3, it can be shown that the characteristic function for the time to complete the composite activity is given by:

$$M_{T_C}(\omega) = \frac{(1-p_B)\ M_{T_A}(\omega)M_{T'_{oB}}(\omega)}{1-p_B M_{T'_{fB}}(\omega)M_{T_{rB}}(\omega)M_{T_A}(\omega)} \tag{4-3}$$

Note the similarity between this result and equation 3-22.

5. Concurrent Activities

A second type of combination is the concurrent carrying out of several activities. In the development which follows, two concurrent activities are considered. Since each of these may in turn be a combination of two or more concurrent activities, no generality is lost.

Activities A and B are initiated simultaneously when their common predecessor activity is completed. Their common successor activity is not initiated until both activities A and B have been completed. The composite activity C is defined as the completion of both activities A and B.

5.1 Independent Recycle Policies

The simplest combination of concurrent activities occurs when their recycle policies are independent. A failure in either activity affects only that activity; the other activity continues undisturbed. This is represented below, in Figure 5-1:

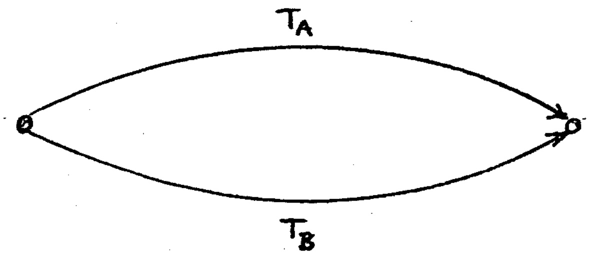

Figure 5-1

The time to complete the composite activity is therefore equal to:

$$T_C = \max (T_A, T_B) \tag{5-1}$$

If the distribution functions for T_A and T_B are available explicitly, then the distribution function for T_C may be found directly. For T_C to be less than some number t_C, both T_A and T_B must be less than t_C. Therefore:

$$F_{T_C}(t_C) = F_{T_A}(t_C)F_{T_B}(t_C) \tag{5-2}$$

and:

$$f_{T_C}(t_C) = f_{T_A}(t_C)F_{T_B}(t_C) + F_{T_A}(t_C)f_{T_B}(t_C) \tag{5-3}$$

It is possible, however, that the distribution functions for T_A and T_B may not be available explicitly. Activities A and B may themselves be composites, and, as the results of prior manipulations, the characteristic functions for T_A and T_B may be given. The distribution functions could be obtained by taking the inverse transform, and the distribution function for T_C may be found in the manner described above. It may sometimes be more convenient, however, to obtain the characteristic function for T_C directly from the characteristic functions for T_A and T_B. From equation 5-2:

$$F_{T_C}(t_C) = F_{T_A}(t_C)F_{T_B}(t_C)$$

Taking the Fourier transform of both sides yields:

$$\frac{M_{T_C}(\omega)}{j\omega} = \frac{1}{2\pi} \frac{M_{T_A}(\omega)}{j\omega} * \frac{M_{T_B}(\omega)}{j\omega}$$

where the asterisk denotes convolution. Therefore:

$$M_{T_C}(\omega) = \frac{\omega}{2\pi j} \int_{-\infty}^{\infty} \frac{M_{T_A}(x)M_{T_B}(\omega-x)}{x\ (\omega-x)}\, dx \qquad (5\text{-}4)$$

5.2 Semi-Dependent Recycle Policies

The recycle policies for the two activities may be semi-dependent in the following sense. A failure in activity A affects both activities, while a failure in activity B affects only that activity. There are two possible semi-dependent recycle policies. A failure in activity A may require that both activities be held during repair and then continued, or it may require that both activities be reinitiated after the repairs have been completed. These will both be discussed.

5.2.1 "Hold and Continue" Policy

When a failure in activity A requires that both activities be held during repairs and then continued, while a failure in activity B does not interrupt activity A, the time required to complete the composite is equal to:

$$T_C = T_{RA} + \max\left(T_{oA}, T_B\right) \qquad (5\text{-}5)$$

where T_{RA} is the sum of all the repair times T_{rA}. By an extension of equation 5-4, the characteristic function for T_C is given by:

$$M_{T_C}(\omega) = \frac{\omega M_{T_{RA}}(\omega)}{2\pi j} \int_{-\infty}^{\infty} \frac{M_{T_{oA}}(x) M_{T_B}(\omega - x)}{x(\omega - x)} \, dx \qquad (5\text{-}6)$$

5.2.2 "Reinitiate" Policy

A failure in activity A may require that both activities be reinitiated; when a failure occurs in activity B, activity A continues undisturbed. This process may be represented by the flow graph shown below, in Figure 5-2:

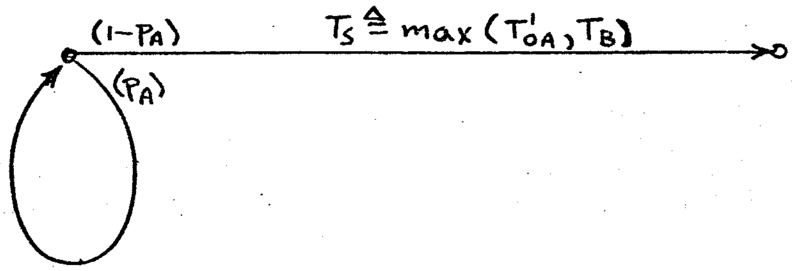

Figure 5-2

From the derivation of Section 5.1, resulting in equation 5-4:

$$M_{T_S}(\omega) = \frac{\omega}{2\pi j}\int_{-\infty}^{\infty} \frac{M_{T'_{oA}}(x)M_{T_B}(\omega-x)}{x\ (\omega-x)}\ dx \qquad (5\text{-}7)$$

and, from the results of Section 3, equation 3-22, the characteristic function for the time to complete the composite is equal to:

$$M_{T_C}(\omega) = \frac{(1-p_A)M_{T_S}(\omega)}{1-p_A M_{T'_{fA}}(\omega)M_{T_{rA}}(\omega)} \qquad (5\text{-}8)$$

5.3 Dependent Recycle Policies

The recycle policies for the two activities may be dependent in the sense that a failure in either activity affects both activities. There are three possible combinations of recycle policies. Both may be "hold and continue" policies, both may be "reinitiate" policies, or one activity may have a "hold and continue" policy and the other a "reinitiate" policy. These three combinations will now be investigated.

5.3.1 "Hold and Continue" Policies

When a failure in either activity requires that both activities be held during repairs and then continued, the total time for the composite is equal to:

$$T_C = T_{RA} + T_{RB} + T_m \qquad (5\text{-}9)$$

where T_m is defined as:

$$T_m \triangleq \max\ (T_{oA}, T_{oB}) \qquad (5\text{-}10)$$

From the derivation of Section 5.1, resulting in equation 5-4:

$$M_{T_m}(\omega) = \frac{\omega}{2\pi j} \int_{-\infty}^{\infty} \frac{M_{T_{oA}}(x)M_{T_{oB}}(\omega-x)}{x(\omega-x)} dx \qquad (5\text{-}11)$$

The characteristic function for T_C is equal to:

$$M_{T_C}(\omega) = M_{T_{RA}}(\omega)M_{T_{RB}}(\omega)M_{T_m}(\omega) \qquad (5\text{-}12)$$

5.3.2 "Reinitiate" Policies

A failure in either activity may require that both be reinitiated after repairs are completed. With such a recycle policy there are two possible modes of failure requiring re-initiation. If a failure occurs in activity A, and no failure has yet occurred in activity B, then both activities are stopped. So far, a time equal to T'_{fA} has elapsed. The equipment is repaired, requiring a time equal to T_{rA}, and both activities are reinitiated. Similarly, if a failure occurs in activity B, and one has not yet occurred in activity A, a time equal to T'_{fB} + T_{rB} is used up and both activities are reinitiated. The first, or "A" failure mode occurs when $T_{fA}<T_{oA}$ and either:

1. $T_{oB} < T_{fB}$
2. $T_{oB} > T_{fB}$ but $T_{fA} < T_{fB}$

The probability of failing in this mode is thus equal to:

$$p'_A = p_A [(1-p_B) + p_B\, p_{AB}] \qquad (5\text{-}13)$$

where p_{AB} is defined as:

$$p_{AB} \triangleq \Pr\left[T_{fA} < T_{fB}\right] \tag{5-14}$$

This is equal to:

$$p_{AB} = \int_{o}^{\infty} F_{T_{fA}}(t_{f_B})dF_{T_{fB}}(t_{f_B}) = \int_{o}^{\infty} F_{T_{fA}}(t_{f_B})f_{T_{fB}}(t_{f_B})dt_{f_B} \tag{5-15}$$

Similarly, the probability of failing in the "B" mode is equal to:

$$p_B' = p_B\left[(1-p_A) + p_A p_{BA}\right] \tag{5-16}$$

where p_{BA} is defined as:

$$p_{BA} = \Pr\left[T_{fB} < T_{fA}\right] \tag{5-17}$$

and is equal to:

$$p_{BA} = \int_{o}^{\infty} F_{T_{fB}}(t_{fA})dF_{T_{fA}}(t_{fA}) = \int_{o}^{\infty} F_{T_{fB}}(t_{fA})f_{t_{fA}}(t_{fA})dt_{fA} \tag{5-18}$$

Note that:

$$p_{AB} + p_{BA} = 1$$

For any one attempt, the probability of successfully performing the activity is equal to:

$$\Pr\,[\text{success}] = 1 - p_A' - p_B'$$

and this is also equal to:

$$\Pr\,[\text{success}] = (1-p_A)(1-p_B)$$

The parallel combination of two activities with dependent recycle policies may be represented by the flow graph shown below, in Figure 5-3:

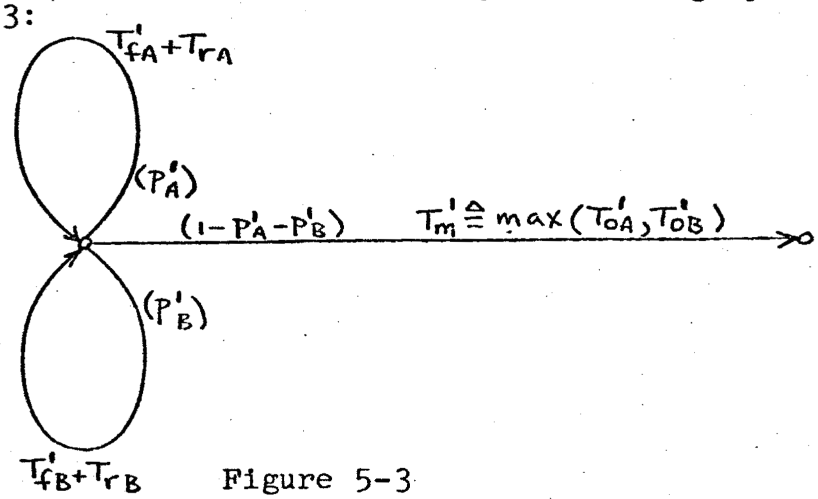

Figure 5-3

Taking into account all possible numbers of failures, in each of the modes "A" and "B", with their respective probabilities of occurrence, $f_{T_C}(t_C)$ is equal to:

$$f_{T_C}(t_C) = \left\{ (1-p_A'-p_B') \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} (p_A')^i (p_B')^j f_{T_m'}(t_C) * [\text{i-fold convolution of } f_{T_{fA}'}(t_C) * f_{T_{rA}}(t_C) \text{ with itself}] * [\text{j-fold convolution of } f_{T_{fB}'}(t_C) * f_{T_{rB}}(t_C) \text{ with itself}] \right\}$$

Taking the Fourier transform of both sides with respect to t_C yields the characteristic function for T_C:

$$M_{T_C}(\omega) = (1-p_A'-p_B') \sum_{i=o}^{\infty} \sum_{j=o}^{\infty} M_{T_m'}(\omega) [p_A' M_{T_{fA}'}(\omega) M_{T_{rA}}(\omega)]^i [p_B' M_{T_{fB}'}(\omega) M_{T_{rB}}(\omega)]^j$$

$$M_{T_C}(\omega) = \frac{(1-p_A'-p_B')\ M_{T_m'}(\omega)}{[1-p_A' M_{T_{fA}'}(\omega) M_{T_{rA}}(\omega)][1-p_B' M_{T_{fB}'}(\omega) M_{T_{rB}}(\omega)]} \tag{5-19}$$

5.3.3 Mixed Recycle Policies

The recycle policies for the two activities may be mixed, in that a failure in activity A requires that both activities be reinitiated after repairs have been completed, while a failure in activity B requires that both activities be held during repairs and then continued. This process may be represented by the flow graph shown below, in Figure 5-4:

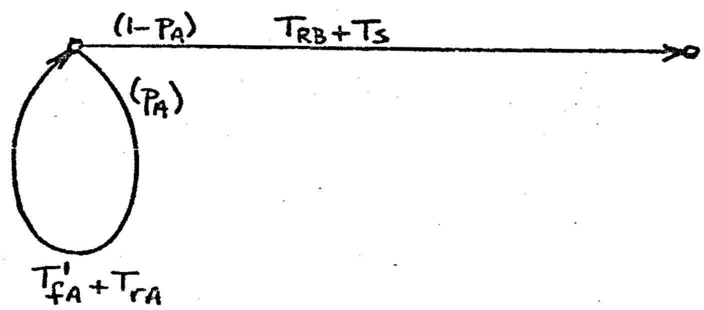

Figure 5-4

where, as in Section 5.2.2, T_S is defined as:

$$T_S \triangleq \max\,(T'_{oA}, T_{oB})$$

From the results of Section 3, equation 3-22, the characteristic function for the time to complete the composite is equal to:

$$M_{T_C}(\omega) = \frac{(1-p_A)M_{T_{RB}}(\omega)M_{T_S}(\omega)}{1-p_A M_{T'_{fA}}(\omega)M_{T_{rA}}(\omega)} \tag{5-20}$$

6. Non-Series-Parallel Combinations

In Section 4, consideration was given to activities which are done serially, in Section 5 to activities performed in parallel. There exists, in addition, a class of combinations which are neither serial nor parallel. An example of such a combination is represented below, in flow graph form in Figure 6-1:

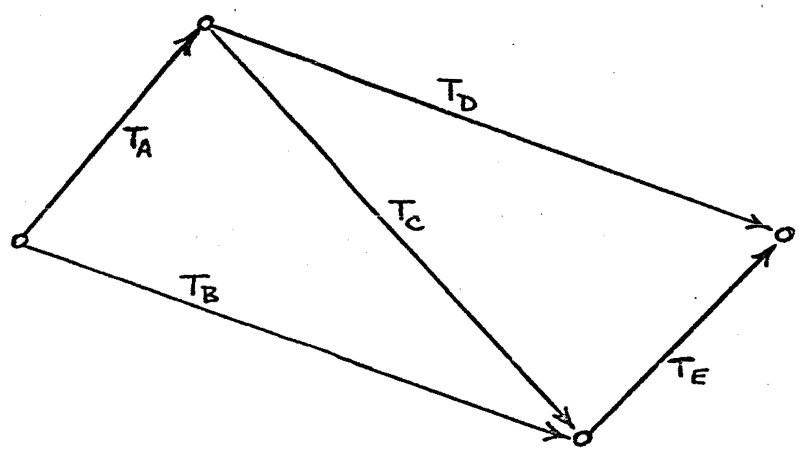

Figure 6-1

Activities A and B are initiated simultaneously. At the completion of activity A, activity D is initiated. At the completion of both activities B and C, activity E is initiated. The composite activity is completed when both activities D and E are completed. The time to complete the composite activity is then equal to:

$$T_M = \max\,[T_A + T_D, T_B + T_E, T_A + T_C + T_E] \tag{6-1}$$

The distribution function for T_M cannot be derived in the manner previously used for the maximum of a set of random variables, since the variables are no longer independent. Another procedure, presented below, must be employed.

The combination of five activities described above is only one type of non-series-parallel combination. By including additional activities, an infinite number of such combinations may be formed. It is not possible, in a paper of finite length, to discuss all of these, nor is it necessary. They all may be handled by extensions of the same procedure. It is sufficient to discuss one such case, the one presented above, and show how the distribution function for T_M may be derived.

Let a new set of five random variables be defined by:

$$\left.\begin{aligned} T_1 &= T_A + T_D \\ T_2 &= T_B + T_E \\ T_3 &= T_A + T_C + T_E \\ T_4 &= T_A \\ T_5 &= T_E \end{aligned}\right\} \qquad (6\text{-}2)$$

The time to complete the composite activity is then equal to:

$$T_M = \max\,[T_1, T_2, T_3] \qquad (6\text{-}3)$$

and the distribution function for T_M is equal to the joint distribution function for $T_1, T_2,$ and T_3:

$$F_{T_M}(t_M) = F_{T_1,T_2,T_3}(t_M, t_M, t_M) \qquad (6\text{-}4)$$

The original random variables, in terms of the new ones, are equal to:

$$\left.\begin{aligned} T_A &= T_4 \\ T_B &= T_2 - T_5 \\ T_C &= T_3 - T_4 - T_5 \\ T_D &= T_1 - T_4 \\ T_E &= T_5 \end{aligned}\right\} \qquad (6\text{-}5)$$

and the Jacobian:

$$J = \begin{Vmatrix} \frac{\partial T_A}{\partial T_1} & \frac{\partial T_A}{\partial T_2} & \frac{\partial T_A}{\partial T_3} & \frac{\partial T_A}{\partial T_4} & \frac{\partial T_A}{\partial T_5} \\ \frac{\partial T_B}{\partial T_1} & \frac{\partial T_B}{\partial T_2} & \frac{\partial T_B}{\partial T_3} & \frac{\partial T_B}{\partial T_4} & \frac{\partial T_B}{\partial T_5} \\ \frac{\partial T_C}{\partial T_1} & \frac{\partial T_C}{\partial T_2} & \frac{\partial T_C}{\partial T_3} & \frac{\partial T_C}{\partial T_4} & \frac{\partial T_C}{\partial T_5} \\ \frac{\partial T_D}{\partial T_1} & \frac{\partial T_D}{\partial T_2} & \frac{\partial T_D}{\partial T_3} & \frac{\partial T_D}{\partial T_4} & \frac{\partial T_D}{\partial T_5} \\ \frac{\partial T_E}{\partial T_1} & \frac{\partial T_E}{\partial T_2} & \frac{\partial T_E}{\partial T_3} & \frac{\partial T_E}{\partial T_4} & \frac{\partial T_E}{\partial T_5} \end{Vmatrix}$$

is equal to 1. Therefore:

$$f_{T_1,T_2,T_3,T_4,T_5}(t_1,t_2,t_3,t_4,t_5) = f_{T_A,T_B,T_C,T_D,T_E,}(t_A,t_B,t_C,t_D,t_E)$$

$$f_{T_1,T_2,T_3,T_4,T_5}(t_1,t_2,t_3,t_4,t_5) = f_{T_A}(t_A)f_{T_B}(t_B)f_{T_C}(t_C)f_{T_D}(t_D)f_{T_E}(t_E)$$

$$f_{T_1,T_2,T_3,T_4,T_5}(t_1,t_2,t_3,t_4,t_5) = f_{T_A}(t_4)f_{T_B}(t_2-t_5)f_{T_C}(t_3-t_4-t_5)$$

$$f_{T_D}(t_1-t_4)f_{T_E}(t_5) \qquad (6\text{-}6)$$

Integrating equation 6-6 over the entire range of t_4 and t_5, and the range of $t_1, t_2,$ and t_3 less than t_M, yields the joint distribution function for $T_1, T_2,$ and T_3, evaluated at t_M:

$$F_{T_1,T_2,T_3}(t_M,t_M,t_M) =$$

$$\int_0^{t_M} dt_1 \int_0^{t_M} dt_2 \int_0^{t_M} dt_3 \int_0^{\infty} dt_4 \int_0^{\infty} dt_5 \; f_{T_1,T_2,T_3,T_4,T_5}(t_1,t_2,t_3,t_4,t_5).$$

From equation 6-4, this is equal to $F_{T_M}(t_M)$. Rearranging the order of integration, and substituting equation 6-6 yields;

$$F_{T_M}(t_M) =$$

$$\int_0^{\infty} dt_4 f_{T_A}(t_4) \int_0^{\infty} dt_5 f_{T_E}(t_5) \int_0^{t_M} dt_1 f_{T_D}(t_1-t_4) \int_0^{t_M} dt_2 f_{T_B}(t_2-t_5)$$

$$\int_0^{t_M} dt_3 f_{T_C}(t_3-t_4-t_5) \qquad (6\text{-}7)$$

Recognizing that the density functions are zero for negative arguments and that the cumulative distribution functions are zero when their arguments are zero, yields:

$$F_{T_M}(t_M) = \int_0^{t_M} dt_4 f_{T_A}(t_4) F_{T_D}(t_1 - t_4) \int_0^{t_M - t_4} dt_5 f_{T_E}(t_5) F_{T_B}(t_M - t_5) F_{T_C}(t_M - t_4 - t_5)$$

(6-8)

By means of equation 6-8, the distribution function for T_M may be determined from the distribution or density functions for T_A, T_B, T_C, T_D, and T_E.

7. Conclusions

In the preceding sections, the various ways in which activities may be logically interrelated and the recycle policies which may be employed in case of failure have been considered. For each type of interrelationship and each recycle policy, a technique has been presented for combining several activities into one composite activity. For each such combination, the probability density function for the time to complete the composite activity has been derived. By successively applying these techniques, the network representation of a large program of activities may be reduced. After each such reduction, the number of branches in the network is decreased, with each branch representing the performance of a greater number of activities. The probability density function for the time required to traverse each branch is derived. After several successive reductions, the network is reduced to a single branch, and the time required to traverse that branch is equal to the time required to complete the entire program. The probability density function for this time is thus derived.

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